ON PROPERTIES OF THE LOWER CENTRAL SERIES
OF ASSOCIATIVE ALGEBRAS

NABILAH ABUGHAZALAH AND PAVEL ETINGOF

ABSTRACT. We give an accessible introduction into the theory of lower central series of associative algebras, exhibiting the interplay between algebra, geometry and representation theory that is characteristic for this subject, and discuss some open questions. In particular, we provide shorter and clearer proofs of the main results of this theory. We also discuss some new theoretical and computational results and conjectures on the lower central series of the free algebra in two generators modulo a generic homogeneous relation.

1. INTRODUCTION

Let A be an associative algebra. Let $L_1(A) = A$, and $L_{i+1}(A) = [A, L_i(A)]$ for $i \geq 1$; so,

$$L_1 \supset L_2 \supset \cdots \supset L_m \supset \ldots$$

is the *lower central series* of A regarded as a Lie algebra. In 2006, in the pioneering paper [FS], Feigin and Shoikhet studied L_i in the case when $A = A_n$ is the free algebra in n generators over a field of characteristic zero. More precisely, they studied the successive quotients $B_i := L_i/L_{i+1}$, and discovered that they have a rich structure: starting from $i = 2$, they are finite length modules over the Lie algebra W_n of polynomial vector fields in n variables. This allows one to say a lot about the structure of B_i (for example, compute its Hilbert series in a number of cases, and prove the surprising fact that B_i has polynomial growth for $i \geq 2$).

Since then, the theory of lower central series of associative algebras was developed in a number of papers, e.g., [DKM, EKM, DE, AJ, BoJ, BB, Ke, BJ, BEJKL, JO, CFZ, KL, FX]. In particular, the papers [EKM, Ke, JO, CFZ] studied the ideals $M_i = AL_i$ and their quotients $N_i = M_i/M_{i+1}$, and showed that they have a similar (and in some ways simpler) structure to L_i, B_i.

The main goal of this paper is to give an accessible introduction into the theory of lower central series of associative algebras, exhibiting the interplay between algebra, geometry and representation theory that is

characteristic for this subject, and to discuss some open questions. In particular, we provide shorter and clearer proofs of the main results of this theory, and also discuss some new results.

This paper is based on lecture notes by the first author of the lectures delivered by the second author at MIT in the Fall of 2014. Many of these results were obtained by undergraduate and high school students and their mentors in the MIT research programs UROP, RSI, SPUR, and PRIMES. One of the goals of this paper is to provide a gentle entry for students who want to work in this field.

The organization of the paper is as follows. In Section 2, we discuss preliminaries and give a review of the main results about the lower central series. In Sections 3-10 we provide proofs of these results. Finally, in Section 11, we discuss the case of algebras with relations and give some new results, computational data, and conjectures.

2. Overview of the main results

2.1. **Preliminaries.**

2.1.1. *The lower central series of a Lie algebra.* Let R be a commutative ring (for example, a field).

Let A be a Lie algebra over R. Define a series of Lie ideals in A inductively:

$$L_1(A) = A;$$

$$L_2(A) = [A, L_1(A)] = [A, A];$$

$$L_3(A) = [A, L_2(A)] = [A, [A, A]];$$

$$\vdots$$

$$L_{i+1}(A) = [A, L_i(A)] = \underbrace{[A, [A, [A, [A \ldots [A, A] \ldots]]]]}_{i+1 \text{ times}}$$

where the bracket $[C, D]$ of two R-submodules C and D is defined to be the span of elements $[c, d]$ such that $c \in C$, $d \in D$. This series is

called the *lower central series* of A. We abbreviate $L_i(A)$ as L_i. We have

$$A = L_1 \supset L_2 \supset L_3 \supset \ldots.$$

We define the successive quotients of the lower central series by $B_i := L_i/L_{i+1}$.

Since $A = L_1 \supset L_2 \supset \ldots$ is a Lie algebra filtration, the direct sum $B := B_1 \oplus B_2 \oplus B_3 \oplus \ldots$ is a graded Lie algebra, i.e. $[B_i, B_j] \subset B_{i+j}$. Moreover, it is easy to check that the Lie algebra B is generated by B_1.

2.1.2. *The lower central series of an associative algebra.* Now let A be an associative unital algebra over R. In the future, we will drop the word "unital" and say just "associative algebra" or "algebra". Define a bracket operation on A by $[a, b] = a \cdot b - b \cdot a$. This operation makes A into a Lie algebra over R. This allows us to define the lower central series $L_i = L_i(A)$ and its successive quotients B_i.

Note that

$$B_1 = A/[A, A] = HH_0(A) = HC_0(A),$$

(the zeroth Hochschild and cyclic homology of A). So B_i for $i \geq 2$ may be viewed as some higher analogues of this.

Denote the two-sided ideals generated by each L_i by M_i, i.e. $M_i := A \cdot L_i \cdot A$. They also form a filtration

$$A = M_1 \supset M_2 \supset M_3 \supset \ldots.$$

It is easy to check that $M_i = A \cdot L_i$.

The motivation for considering M_i is that A/M_i is the maximal quotient of A which is Lie nilpotent of nilpotency degree i, i.e

$$[\ldots [a_1, a_2], \ldots, a_i] = 0 \quad \forall a_1, a_2 \ldots, a_i \in A/M_i.$$

This is an important special case of a polynomial identity in an algebra.

We also define the successive quotients $N_i := M_i/M_{i+1}$.

For example, $N_1 = A_{\mathrm{ab}}$, the abelianization of the algebra A, obtained by taking the quotient of A by the relation $[a, b] = 0$. The R-modules B_i and N_i will be the main objects of study in this paper.

Example 2.1. 1. Let $A = A_n = A_n(R) = R < x_1, x_2, \ldots, x_n >$ be the free non-commutative algebra in n generators. A_n is a free R-module, with an R-basis formed by all possible monomials, or words, in the letters $x_1, \ldots, x_n$. So, we have

$$\bigcap_{i \geq 1} L_i = \bigcap_{i \geq 1} M_i = 0.$$

Indeed, L_i, M_i are graded by length of words and so when we intersect, the minimal possible degree goes to infinity. So, the spaces $B = \oplus_{i \geq 1} B_i$

and $N = \oplus_{i \geq 1} N_i$ can serve as a "graded approximation" to A. This will be one of the main examples in this paper.

2. Let $f_1, ..., f_m$ be homogeneous elements in A_n. Let

$$A = A_n / < f_1, f_2, \ldots, f_m > .$$

Then $\bigcap_{i \geq 1} L_i = \bigcap_{i \geq 1} M_i = 0$ for the same reason as above.

3. If A is a commutative algebra then $M_i = L_i = N_i = B_i = 0$ for $i \geq 2$.

4. If $A = \mathbb{C} < x, y > / < yx - xy - 1 >$ (the Weyl algebra) then $A = [A, A]$ (as $x^i y^j = [y, \frac{x^{i+1} y^j}{i+1}]$), so $L_i = A$ and $B_i = N_i = 0$ for all i.

Hence, the last two examples will not be interesting for us in this paper.

Definition 2.2. Let $M = \bigoplus_{i \geq 0} M[i]$ be a graded $R-$module such that $\mathrm{rank}_R(M[i]) < \infty$. The Hilbert series of M is defined to be

$$h_M(t) = \sum_{i=0}^{\infty} \mathrm{rank}_R(M[i]) t^i.$$

For instance, if R is a field, we have

$$h_M(t) = \sum_{i=0}^{\infty} \dim(M[i]) t^i.$$

Example 2.3. If $M = A_n$ graded by length of words then $\dim(M[i]) = n^i$, so $h_M(t) = \frac{1}{1-nt}$.

2.2. **Results on the lower central series of general algebras.**

Theorem 2.4. ([J]) *If A is a finitely generated algebra then for all k there exists m such that $M_2^m \subset M_k$. This m depends only on the number of generators of A and on k.*

A proof of Theorem 2.4 over $\mathbb{C}$ will be given in Section 5, see Corollary 5.3 (and essentially the same proof goes through over any base ring).

Theorem 2.5. ([GL]) $M_i \cdot M_j \subset M_{i+j-2}$, *for every $i, j \geq 2$.*

A proof of Theorem 2.5 is given in Section 4.

Definition 2.6. Let I be an ideal of an algebra A. We say that I is:

(1) nil if for all $x \in I$ there exists N such that $x^N = 0$.
(2) nilpotent if there exists N such that $I^N = 0$.

Corollary 2.7. *If A is Lie nilpotent then M_2 is a nil ideal. Moreover, M_2 is nilpotent if A is also finitely generated.*

Proof. Assume that A has nilpotency degree k, i.e. $M_k = 0$. Let $x \in M_2$. Then $x \in M_2(A')$ for some finitely generated subalgebra $A' \subset A$. By Theorem 2.4, there exists m such that $M_2(A')^m \subset M_k(A')$. But $M_k(A') \subset M_k(A) = 0$. Hence $x^m = 0$, and M_2 is nil. If A is in addition finitely generated, then by Theorem 2.4 there is an m such that $M_2^m \subset M_k = 0$, so M_2 is nilpotent. $\qquad\square$

Theorem 2.8 ([BJ])**.** *If i or j is odd and $\frac{1}{6} \in R$ then $M_i M_j \subset M_{i+j-1}$.*

Theorem 2.8 is proved in Section 3.

Conjecture 2.9. If $i, j \geq 2$ are both even then in general $M_i \cdot M_j \not\subset M_{i+j-1}$.

One of the important corollaries of Theorem 2.8 is the following theorem. Suppose A is generated by $x_1, ..., x_n$, and let $A_{\leq 2}$ be the span of elements $1, x_i, x_i x_j$.

Theorem 2.10. *([AJ]) Let A be an algebra over a field of characteristic zero. For $m \geq 2$ we have*

$$B_{m+1} = [A_{\leq 2}, B_m] + \sum_{i,j,k \ distinct} [x_i[x_j, x_k], B_m].$$

This theorem is proved in Section 9.

Remark 2.11. Later Bapat and Jordan showed that the last summand is in fact redundant, i.e. one has the following stronger (but more difficult) theorem (conjectured in [AJ]).

Theorem 2.12. *([BJ]) Retain the assumptions of Theorem 2.10. Then for $m \geq 2$, one has $B_{m+1} = [A_{\leq 2}, B_m]$.*

We refer the reader to [BJ] for a proof of this result.

2.3. **Results on the lower central series of the free algebra $A_n(\mathbb{C})$.** Let $A = A_n$ and $R = \mathbb{C}$. We are interested in the Hilbert series of N_i and B_i.

First consider the case $i = 1$. Recall that $N_1 = A_{ab} = \mathbb{C}[x_1, x_2, \ldots, x_n]$, so $h_{N_1}(t) = \frac{1}{(1-t)^n}$. Also, recall that $B_1 = A/[A, A]$. Therefore, it has a basis consisting of cyclic words (necklaces) in $x_1, x_2, \ldots, x_n$, i.e. words considered up to cyclic permutation. So, $\dim B_1[d]$ is equal to the number of necklaces of length d. Let us denote this number by $a_d(n)$. It follows from Polya's enumeration theorem that

$$\prod_{d \geq 1}(1 - t^d)^{a_d(n)} = (1 - nt)(1 - nt^2)(1 - nt^3) \ldots,$$

which allows us to easily compute $a_d(n)$ recursively. Hence, $a_d \sim n^d$ (in sense that $\lim_{d \to \infty} \frac{\log(a_d(n))}{d} = \log n$), and in particular $a_d(n)$ has exponential growth as $d \to \infty$.

It turns out, however, that the dimensions of the homogeneous parts $B_i[d]$ of the spaces B_i grow polynomially for all $i \geq 2$ (and the same holds for N_i). Namely, the spaces B_2, N_2 are known explicitly, and we'll discuss their structure and Hilbert series below. The Hilbert series of B_i, N_i for $i \geq 3$ are known only in a few special cases, but we have the following theorem.

Theorem 2.13 ([FS, DE, AJ]). *The Hilbert series $h_{B_i}(t)$ and $h_{N_i}(t)$ for $i \geq 3$ are of the form $\frac{P(t)}{(1-t)^n}$, where $P(t)$ is a polynomial with positive integer coefficients.*

Theorem 2.13 is proved in Section 10.

As we mentioned, this property fails for B_1, but this is "corrected" by the following proposition.

Proposition 2.14. *([FS]) Let A be any algebra over $\mathbb{C}$, $Z \subset B_1 = A/[A, A]$ be the image of M_3 in B_1, i.e. $Z = M_3/M_3 \cap L_2$, and $\overline{B_1} = B_1/Z$. Then*

(1) $Z \subset B_1$ is central in B (so that $\overline{B} := B/Z = \overline{B_1} \oplus B_2 \oplus ...$ is a graded Lie algebra);

(2) for $A = A_n$, we have $h_{\overline{B_1}}(t) = \frac{P(t)}{(1-t)^n}$, where $P(t)$ is a polynomial.

Proposition 2.14(1) is proved in Section 4. Proposition 2.14(2) follows from Theorem 2.17(5) below.

Now let us describe B_2 and N_2 in the case of the free algebra $A = A_n(\mathbb{C})$. We will get this description from the explicit description of A/M_3, obtained by Feigin and Shoikhet ([FS]). Namely, let $\Omega = \Omega(\mathbb{C}^n)$ be the space of differential forms on $\mathbb{C}^n$ with polynomial coefficients. We have a decomposition $\Omega = \oplus_{k=0}^n \Omega^k$, where Ω^k is the space of differential forms of degree k, which is a free module over $\mathbb{C}[x_1, ..., x_n]$ with basis $dx_{i_1} \wedge ... \wedge dx_{i_k}$, $i_1 < ... < i_k$. Let $\Omega^{\text{even}} = \oplus_{0 \leq i \leq n/2} \Omega^{2i}$ be the space of even forms. The space Ω is a supercommutative algebra under wedge product, and Ω^{even} is a commutative subalgebra in it. Also, Ω carries a de Rham differential $d : \Omega^k \to \Omega^{k+1}$.

Following Feigin and Shoikhet [FS], we introduce another product on Ω (and Ω^{even}), denoted $*$. Namely, we set

$$\alpha * \beta = \alpha \wedge \beta + \frac{1}{2} d\alpha \wedge d\beta;$$

the corresponding commutator on Ω_{even} is given by

$$[\alpha, \beta]_* = \alpha * \beta - \beta * \alpha = d\alpha \wedge d\beta,$$

if $\alpha,\ \beta\ \in\ \Omega_{\text{even}}$ (below we will denote this commutator just by $[,]$ when no confusion is possible). It is easy to check that this product is associative, but, unlike the wedge product, it is noncommutative on even forms, and does not preserve the grading by degree of differential forms. We will denote the algebra Ω with this operation by Ω_*, and its even part by Ω_*^{even}. Note that $\Omega = \Omega_*$ as vector spaces, and we will use these two notations interchangeably when we don't consider multiplication.

Proposition 2.15. *The algebra* Ω_*^{even} *satisfies the polynomial identity* $[[a, b], c] = 0$.

Proof. Since
$$[a, b] = da \wedge db = d(a \wedge db),$$
we have $d[a, b] = 0$, and hence
$$[[a, b], c] = d[a, b] \wedge dc = 0.$$

$\square$

Corollary 2.16. *The algebra homomorphism* $\widetilde{\phi} : A \to \Omega_*^{\text{even}}$ *defined by* $\widetilde{\phi}(x_i) = x_i$ *descends to a homomorphism* $\phi : A/M_3 \to \Omega_*^{\text{even}}$.

Proof. This immediately follows from Proposition 2.15. $\square$

One of the main theorems about lower central series of associative algebras is the following theorem about the lower central series of $A_n(\mathbb{C})$.

Theorem 2.17. *([FS])*
  (1) ϕ *is an algebra isomorphism.*
  (2) $\phi(N_2) = \Omega^{\text{even},+} = \oplus_{1 \leq i \leq n/2} \Omega^{2i}$, *the space of even forms of positive degree.*
  (3) *The image of* L_2 *in* A/M_3 *is naturally isomorphic to* B_2 *(i.e.* $L_2 \cap M_3 = L_3$*).*
  (4) $\phi(B_2) = \Omega_{\text{exact}}^{\text{even}}$, *the space of even exact forms, i.e. of even forms* α *such that* $d\alpha = 0$.
  (5) ϕ *induces a map* $\hat{\phi} : B_1 \to \Omega^{\text{even}}/\Omega_{\text{exact}}^{\text{even}}$ *such that* $\hat{\phi}(Z) = 0$, *so* $\hat{\phi}$ *defines a map* $\overline{\phi} : \overline{B}_1 \to \Omega^{\text{even}}/\Omega_{\text{exact}}^{\text{even}}$. *The map* $\overline{\phi}$ *is an isomorphism.*

Parts (1),(2),(4) and (5) of Theorem 2.17 are proved in Section 6, and part (3) (which is the hardest one) is proved in Section 8.

Using Theorem 2.17, it is easy to compute the Hilbert series of $\overline{B}_1$, B_2 and N_2 for $A = A_n$.

Let us now present some bounds on the degree of the polynomial $P(t)$ in the numerator of the Hilbert series of B_m, N_m for $m \geq 3$ (see Theorem 2.13).

Theorem 2.18 ([AJ]). *For $m \geq 3$ we have*

$$deg\big(h_{B_m}(t)(1-t)^n\big) \leq 2m - 3 + 2\lfloor\frac{n-2}{2}\rfloor;$$

$$deg\big(h_{N_m}(t)(1-t)^n\big) \leq 2m - 2 + 2\lfloor\frac{n-2}{2}\rfloor.$$

These bounds are proved using representation theory. Namely, let W_n be the Lie algebra of polynomial vector fields on $\mathbb{C}^n$. In more detail, W_n consists of expressions of the form $v = \sum_i f_i(x_1, ..., x_n)\partial_n$, where $f_i \in \mathbb{C}[x_1, ..., x_n]$, with commutator defined by

$$[\sum_i f_i\partial_i, \sum_j g_j\partial_j] = \sum_{i,j}(f_i\frac{\partial g_j}{\partial x_i}\partial_j - g_j\frac{\partial f_i}{\partial x_j}\partial_i).$$

Theorem 2.19.   (1) ([FS]) $\overline{B}_1(A_n)$ *and* $B_i(A_n)$, $i \geq 2$ *are natu-rally graded modules over the Lie algebra* W_n.
   (2) ([EKM]) $N_i(A_n)$ *are naturally graded modules over* W_n.

Let λ be a partition with at most n parts. To this partition one can attach the polynomial $GL(n)$-module V_λ, and we can define the module $\mathcal{F}_\lambda := V_\lambda \otimes \mathbb{C}[x_1, ..., x_n]$ over W_n of tensor fields on $\mathbb{C}^n$ of type λ (see Example 7.2 below). Note that $\mathcal{F}_\lambda$ is graded by the eigenvalues of the Euler field $E := \sum_i x_i\frac{\partial}{\partial x_i}$, and

$$(1) \qquad\qquad h_{\mathcal{F}_\lambda}(t) = \frac{\dim V_\lambda t^{|\lambda|}}{(1-t)^n},$$

We will see below in Theorem 7.5 that $\mathcal{F}_\lambda$ is irreducible if and only if $\lambda_1 \geq 2$, or $\lambda = (1^n)$.

The following result is proved in the papers [AJ, BJ, Ke]:

Theorem 2.20. *For all $m \geq 3$, B_m and N_m are of finite length as modules over W_n, with all composition factors being $\mathcal{F}_\lambda$ with $|\lambda| \geq 2$. Moreover, if $\mathcal{F}_\lambda$ occurs in B_m then $|\lambda| \leq 2m - 3 + 2\lfloor\frac{n-2}{2}\rfloor$, and if $\mathcal{F}_\lambda$ occurs in N_m then $|\lambda| \leq 2m - 2 + 2\lfloor\frac{n-2}{2}\rfloor$.*

This theorem together with formula (1) implies Theorem 2.18.

We also have the following interesting result about arbitrary finitely generated algebras A over a field, proved independently by Jordan and Orem [JO] and by Cordwell, Fei, and Zhu [CFZ]:

Theorem 2.21. *Suppose that $\mathrm{Spec}(A_{ab})$ is at most 1-dimensional and has finitely many non-reduced points. Then B_m and N_m for $m \geq 2$ are finite dimensional.*

For the proof, we refer the reader to [JO, CFZ].

3. Proof of Theorem 2.8

We will now prove Theorem 2.8. To this end, we introduce some lemmas, which can be found in the paper [BJ].

For brevity introduce the notation

$$[a_1, a_2, \ldots, a_m] := [a_1, [a_2, \ldots [a_{m-1}, a_m]] \ldots].$$

Lemma 3.1. *If* $1/6 \in R$ *then* $[M_3, A] \subset L_4$.

Proof. Let $[x[y, z, u], v] \in [M_3, A]$ for $A = A_5$ which is generated by x, y, z, u, v. Our job is to show that $[x[y, z, u], v] \in L_4$. First note that $[x[y, z, u], v] = [x * [y, z, u], v]$ modulo L_5 where $a * b = \frac{1}{2}(ab + ba)$. Then we have the following:

$$(2) \qquad [x * [y, z, u], v] + [y * [x, z, u], v] = [[x * y, z, u], v] \in L_4;$$

$$(3) \qquad [x * [y, z, u], v] + [v * [y, z, u], x] = -[x * v, y, z, u] \in L_4.$$

This means that the expression $[x * [y, z, u], v]$ is anti-symmetric under the group $G = S_3\{x, y, v\} \times S_2\{z, u\}$ as an element of A/L_4. Now we have the following identity, which may be verified directly on coefficients of the 10 monomials in x, y, z, u, v which represent G-orbits on all the monomials having degree 1 in each variable:

$$(4) \qquad \text{Alt}_G[x * [y, z, u], v] = \text{Alt}_G(4[z * x, y, v, u] - 2[x, z, y, u * v]),$$

where Alt_G is antisymmetrization with respect to G. So we conclude that $[x[y, z, u], v] \in L_4$. $\square$

Lemma 3.2. *If* $1/6 \in R$ *then* $[M_3, L_k] \subset L_{k+3}$.

Proof. Consider $[a[b, c, d], [x, y]]$, where $x \in L_1 = A$, $y \in L_{k-1}$. We have

$$(5) \qquad [a[b, c, d], [x, y]] = [[a[b, c, d], x], y] + [x, [a[b, c, d], y]].$$

We prove the statement by induction on k. The statement is true when $k = 1$ by Lemma 3.1. Now consider $k > 1$ and assume that the statement is true for $k - 1$. Notice that the first term in the RHS of (5) is in L_{k+3} by Lemma 3.1 and the second term is in L_{k+3} by the inductive hypothesis. Therefore, $[a[b, c, d], [x, y]] \in L_{k+3}$ as required. $\square$

The following lemma is standard and follows easily from the Jacobi identity.

Lemma 3.3. *Any polylinear element of the free Lie algebra in* N *generators* $w_1, w_2, \ldots, w_N$ *(i.e. one of degree 1 in every generator) can be written as a linear combination of*

$$[w_{\sigma(1)}, \ldots, w_{\sigma(N-1)}, w_N], \sigma \in S_{N-1}.$$

Corollary 3.4. *A polylinear element of $L_{N-1}(A_N)$ is a linear combination of $N-1$-tuple commutators, where all entries are single letters, except the last one, which is a product of two letters.*

Proposition 3.5. *If $\frac{1}{6} \in R$ then $[M_j, L_k] \subset L_{k+j}$ whenever j is odd.*

Proof. The proof is by induction in j, running over odd values of j. For $j = 1$ the statement is trivial, so we just have to justify the induction step, from $j-2$ to j. Consider a general element $[m, l]$ with $m \in M_j$ and $l = [l_1, l_2, \ldots, l_k] \in L_k$. Write $m = a[b, c, d]$, $d \in L_{j-2}$, $a, b, c \in A$. Let us regard $a, b, c, d, l_1, ..., l_k$ as independent variables (in particular, we regard d as a variable instead of an iterated bracket). Then by Lemma 3.2, $[a[b, c, d], l]$ is a polylinear element of $L_{k+3}(A_{k+4})$, where A_{k+4} is the free algebra generated by $a, b, c, d, l_1, l_2, \ldots, l_k$. By Corollary 3.4, this element is a linear combination of terms of the form
(6)
$$[y_1, \ldots, y_{k+2}, y_{k+3}d], \quad [y_1, \ldots, y_{k+2}, dy_{k+3}], \quad [y_1, \ldots, d, \ldots, y_{k+1}, y_{k+2}y_{k+3}]$$
for various permutations $(y_1, \ldots, y_{k+3})$ of $(a, b, c, l_1, \ldots, l_k)$. We now plug in $d = [d_1, \ldots, d_{j-2}]$. Then terms of the third type in (6) lie in L_{k+j} by definition. Also, by the induction assumption $[y_{k+2}, y_{k+3}d] \in L_{j-1}$, hence $[y_{k+2}, dy_{k+3}] \in L_{j-1}$. Plugging this into the terms of the first two types in (6), we find that they are also in L_{k+j}. This justifies the step of induction. □

Remark 3.6. Here is another version of the proof of Proposition 3.5, due to Darij Grinberg.

Let us consider a more general setting. Let A be an algebra, and D be a Lie ideal of A. We define $L_i(D) \subset A$ for all $i \geq 1$ recursively by $L_1(D) = D$; $L_{i+1}(D) = [A, L_i(D)]$. (So $L_i(D) = [A, [A, ..., [A, D]...]]$, with i being the total number of A's plus 1.) We reserve the notation L_i for $L_i(A)$.

Notice that $L_i(L_j(D)) = L_{i+j-1}(D)$ for all i and j. It is also easy to see that all $L_i(D)$ are Lie ideals of A and satisfy $[L_i(D), L_j] \subseteq L_{i+j}(D)$.

We also set $M_i(D) = AL_i(D) = L_i(D)A$ (the last equality holds because $L_i(D)$ is a Lie ideal of A). Clearly, $M_i(D)$ is an ideal of A.

From now on, we assume that D is an ideal (not just a Lie ideal) of A. ("Ideal" always means "two-sided ideal" here.) Here is a slight generalization of Lemma 3.2:

Lemma 3.7. *If $1/6 \in R$, then $[M_3(D), A] \subseteq L_4(D)$.*

Proof. The proof is the same as the proof of Lemma 3.2, up to adding "(D)"s after the L's and M's. The trick is that every iterated commutator of 4 elements of A lies in D as long as at least one of these 4

elements lies in D, and that D is an ideal under the $*$-product (because it is an ideal under the usual product). $\qquad\square$

Next, we generalize Lemma 3.2:

Lemma 3.8. *If $1/6 \in R$, then $[M_3(D), L_k] \subseteq L_{k+3}(D)$.*

Proof. The proof again is the same as that of Lemma 3.2, using Lemma 3.1, but now d belongs to D. $\qquad\square$

The proof of Proposition 3.5 now proceeds by induction over odd j as before, but the induction step becomes simpler to explain. Namely, let $D = M_{j-2}$. This is clearly an ideal of A. Also, $L_{j-2} \subseteq D$. Now, $M_j = AL_j = A[A, [A, L_{j-2}]] \subseteq A[A, [A, D]]$ (since $L_{j-2} \subseteq D$) $= M_3(D)$, whence $[M_j, L_k] \subseteq [M_3(D), L_k] \subseteq L_{k+3}(D)$ (by Lemma 3.8) $= L_{k+2}([A, D]) \subseteq L_{k+2}(L_{j-1})$ (since $[A, D] = [D, A] = [M_{j-2}, L_1] \subseteq L_{j-1}$ by the induction hypothesis) $= L_{k+j}$.

Proof of Theorem 2.8. We may assume that k is odd. Clearly it is enough to show that $L_j L_k \subset M_{j+k-1}$. Let $a \in A$, $x \in L_{j-1}$, $y \in L_k$. Then we have

$$[x, a]y = [x, ay] - a[x, y].$$

The LHS is a completely general generator of $L_j L_k$. By Proposition 3.5, the first term of the RHS is in M_{j+k-1}. Also, it is clear that the second term of the RHS is in M_{j+k-1}. This implies the theorem. $\qquad\square$

Remark 3.9. A computer calculation with polylinear elements of A_5 shows that Lemma 3.1 holds over $R = \mathbb{Z}[\frac{1}{3}]$. Hence Theorem 2.8 holds when $\frac{1}{3} \in R$, with the same proof. However, as shown by Krasilnikov [Kr], Theorem 2.8 and Lemma 3.1 fail over $\mathbb{Z}$ and over fields of characteristic 3.

4. Proof of Theorem 2.5 and Proposition 2.14(1)

4.1. **Proof of Theorem 2.5.** For the proof, we need the following lemma, which can be proved by using the Leibniz rule.

Lemma 4.1. *One has*

$$[c_1, c_2, \ldots, c_n, ab] = \sum_j \sum_{\sigma \in T_j} [c_{\sigma(1)}, \ldots, c_{\sigma(j)}, a][c_{\sigma(j+1)}, \ldots, c_{\sigma(n)}, b]$$

where T_j is the set of permutations $\sigma \in S_n$ such that $\sigma(1) < \cdots < \sigma(j)$ and $\sigma(j+1) < \cdots < \sigma(n)$.

It suffices to show that $L_j L_k \subset M_{j+k-2}$. The proof is by induction on k. Assume that for any $2 \leq s < k$, $L_j L_s \subset M_{j+s-2}$. So our job

is to show that $[r, a_j][b_1, \ldots, b_k] \in M_{j+k-2}$, where $r = [a_1, \ldots, a_{j-1}]$. Consider the expression

$$T := [b_1, \ldots, b_{k-2}, r, a_j[b_{k-1}, b_k]].$$

By Lemma 4.1, this is the sum of the following expressions:

(1) $[r, a_j][b_1, \ldots, b_k]$.

(2) $[b_{\sigma(1)}, \ldots, b_{\sigma(l)}, r, a_j][b_{\sigma(l+1)}, \ldots, b_{\sigma(k-2)}, b_{k-1}, b_k]$, $l > 0$. This is in M_{j+k-2} by the inductive hypothesis.

(3) $[b_{\sigma(1)}, \ldots, b_{\sigma(l)}, a_j][b_{\sigma(l+1)}, \ldots, b_{\sigma(k-2)}, r, b_{k-1}, b_k]$. This is contained in $L_{l+1} \cdot L_{k+j-l-1}$. Since $l \leq k - 2$, we have $l + 1 < k$, so this is also in M_{k+j-2} by the induction hypothesis for the opposite ring A^{op}.

So it remains to show that $T \in M_{j+k-2}$. But

$$T = -[b_1, \ldots, b_{k-2}, a_j[b_{k-1}, b_k], r] \in L_{j+k-2}.$$

4.2. **Proof of Proposition 2.14(1).**

Proposition 4.2 ([FS])**.** *Over any ring R we have $[M_3, L_j] \subset L_{j+2}$.*

Proof. It can be shown easily that $[A, M_3] \in L_3$ by just checking that $[x[y, z, u], v] \in L_3$ for $A = A_5$ (this can be done by hand or using a computer). Now, using Lemma 3.3, we get $[M_3, L_j] \subset [A...[A, M_3]..]$ (j commutators), and the latter is in L_{j+2}, since $[A, M_3] \subset L_3$. $\square$

Corollary 4.3. *(Proposition 2.14(1)) Let $Z \subset B_1$ be the image of M_3 in B_1. Then Z is central in the graded Lie algebra $B = \bigoplus_{i \geq 1} B_i$.*

Proof. By Proposition 4.2 we have that $[M_3, A] \subset L_3$ and hence we get $[B_1, Z] = 0$, as the image of L_3 in B_2 is 0. But B_1 generates B as a Lie algebra, so $[B, Z] = 0$. $\square$

Therefore, we can define the Lie algebra $\overline{B} = B/Z$, where $\overline{B} = \overline{B_1} \oplus \bigoplus_{i \geq 2} B_i$ and $\overline{B_1} = B_1/Z$.

5. Proof of Theorem 2.17 (1),(2),(4),(5)

5.1. **Proof of Theorem 2.17(1).** Recall that we have a new non-commutative product on differential forms $\Omega = \Omega(\mathbb{C}^n) = \bigoplus_{k \geq 0} \Omega^k(\mathbb{C}^n)$,

given by:

$$\alpha * \beta = \alpha \wedge \beta + \frac{1}{2} d\alpha \wedge d\beta;$$

the corresponding commutator is given by

$$[\alpha, \beta]_* = \alpha * \beta - \beta * \alpha = d\alpha \wedge d\beta,$$

if α, $\beta \in \Omega^{\text{even}}$. We have by Corollary 2.16 a homomorphism $\widetilde{\phi}$: $A \to \Omega_*^{\text{even}}$ defined by $\widetilde{\phi}(x_i) = x_i$, and it descends to a homomorphism $\phi : A/M_3 \to \Omega_*^{\text{even}}$.

Proposition 5.1. *(Theorem 2.17(1)) ϕ is an isomorphism.*

Proof. First we show surjectivity. Note that $\phi([x_i, x_j]) = dx_i \wedge dx_j$. This implies that

$$\phi(x_1^{m_1} \cdots x_n^{m_n}[x_{i_1}, x_{i_2}] \cdots [x_{i_{2r-1}}, x_{i_{2r}}]) =$$

$x_1^{m_1} \cdots x_n^{m_n} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{2r-1}} \wedge dx_{i_{2r}} + \text{higher degree terms}.$
But such elements span Ω_*^{even}, so ϕ is surjective.

Now we show injectivity. Observe that for any algebra A and any $x, y, z, t \in A$, we have modulo M_3:

$$\begin{aligned}
[x, y][z, t] + [x, z][y, t] &= [x, y[z, t]] + [x, z[y, t]] \\
&= [x, [yz, t]] - [x, [y, t]z] + [x, z[y, t]] \\
&= [x, [yz, t]] + [x, [z, [y, t]]] \\
&= 0 \ (\text{mod } M_3).
\end{aligned}$$

In particular, $[x, y][x, t] = 0 \mod M_3$. Hence the elements $u_{ij} = [x_i, x_j] \in A/M_3$ satisfy $u_{ij}u_{il} = 0$, $u_{ij}u_{kl} + u_{kj}u_{il} = 0$ where i, j, k, l are distinct. Also $[x_i, u_{jk}] = 0$, so u_{jk} is central in A/M_3. So the elements $x_1^{m_1} \cdots x_n^{m_n} u_{i_1 i_2} \cdots u_{i_{2r-1} i_{2r}}$, for all possible $m_1, \cdots, m_n \geq 0$ and $1 \leq i_1 < i_2 < i_3 < \cdots < i_{2r} \leq n$ form a spanning set for A/M_3. But the images of these elements under ϕ are linearly independent, so ϕ is injective and thus an isomorphism. $\square$

5.2. **Proof of Theorem 2.17(2).**

Corollary 5.2. *(Theorem 2.17(2)) $\phi(N_2) = \Omega^{\text{even},+} = \oplus_{1 \leq i \leq n/2} \Omega^{2i}$, the space of even forms of positive degree.*

Proof. Note that N_2 is the span of the elements $a[b, c]$; $a, b, c \in A/M_3$. Let $\alpha, \beta, \gamma \in \Omega^{\text{even}}$ and let $a = \phi^{-1}(\alpha)$, $b = \phi^{-1}(\beta)$, $c = \phi^{-1}(\gamma) \in A/M_3$ (this makes sense since ϕ is an isomorphism by Theorem 2.17(1)). Then $\phi([b, c]) = d\beta \wedge d\gamma$, so $\phi(a[b, c]) = \alpha \wedge d\beta \wedge d\gamma$. But $\Omega^{\text{even},+}$ is spanned by such elements. $\square$

Corollary 5.3. *(Theorem 2.4 over $\mathbb{C}$) If A is an algebra over $\mathbb{C}$ with $\leq 2r - 1$ generators then $M_2^r \subset M_3$. Hence $M_2^{(k-2)r} \subset M_k$ for $k \geq 3$.*

Proof. It suffices to assume that $A = A_n$, $n \leq 2r - 1$. Then, by Theorem 2.17(2), $\widetilde{\phi}(M_2) \subset \Omega^{\text{even},+} \subset \Omega^+$. Therefore $\widetilde{\phi}(M_2^r) \subset \Omega^{\geq 2r} = 0$, which implies that $M_2^r \subset M_3$. But by Theorem 2.5, $M_3 M_i \subset M_{i+1}$ for all i, which by induction in k yields $M_2^{(k-2)r} \subset M_k$ for all $k \geq 3$. $\square$

5.3. **Proof of Theorem 2.17(4,5).**

Corollary 5.4. *(Theorem 2.17(4,5)) (i) $\widetilde{\phi}(L_2) = \Omega^{\mathrm{even}}_{\mathrm{exact}} = \{\omega \in \Omega^{\mathrm{even}}, \, \omega = d\alpha\}$.*

(ii) The map $\widetilde{\phi}$ induces a map $\hat{\phi} : B_1 \to \Omega^{\mathrm{even}}/\Omega^{\mathrm{even}}_{\mathrm{exact}}$ which kills Z, so defines a map

$$\overline{\phi} : \overline{B}_1 \to \Omega^{\mathrm{even}}/\Omega^{\mathrm{even}}_{\mathrm{exact}},$$

which is an isomorphism.

Proof. (i) L_2 is spanned by $[a, b]$, $a, b \in A$. We have

$$\widetilde{\phi}([a, b]) = d\widetilde{\phi}(a) \wedge d\widetilde{\phi}(b) = d(\widetilde{\phi}(a) \wedge d\widetilde{\phi}(b))$$

which is exact. Moreover, fix b such that $\phi(b) = w$, a given element of Ω^{even} of degree $2r$. Then $\phi([b, x_i]) = dw \wedge dx_i = d(w \wedge dx_i)$. But elements $w \wedge dx_i$ span the space of odd forms. This implies (i).

(ii) follows immediately from (i). $\qquad\square$

6. PROOF OF THEOREM 2.19

Let A be any algebra and let $\mathrm{Der}(A)$ be the Lie algebra of derivations of A. Let $D \in \mathrm{Der}(A)$ be such that $D(A) \subset M_3$.

Proposition 6.1. *For all $i \geq 2$, $D(L_i) \subset L_{i+1}$.*

Proof. By the Leibniz rule we have

$$D[a_1, a_2, \cdots, a_i] = \sum_{j=1}^{i} [a_1, \cdots, [Da_j, \cdots, a_i]].$$

Using the Jacobi identity, this can be written as a linear combination of elements of the form $[a_{\sigma(1)}, ..., [a_{\sigma(i-1)}, Da_{\sigma(i)}]]$, where $\sigma \in S_i$. But we know that $[a_{\sigma(i-1)}, Da_{\sigma(i)}] \in L_3$ since $[M_3, A] \subset L_3$ by Proposition 4.2. Hence $D[a_1, \cdots, a_i] \in L_{i+1}$. $\qquad\square$

So we get $D|_{B_i} = 0$ for all $i \geq 2$ and $D|_{\overline{B}_1} = 0$. Also we have

$$D(M_i) = D(AL_i) \subset D(A)L_i + AD(L_i),$$

but $D(A)L_i \subset M_3 L_i \subset M_{i+1}$ by Theorem 2.5 and $AD(L_i) \subset AL_{i+1} = M_{i+1}$ by Proposition 6.1. Therefore, $D(M_i) \subset M_{i+1}$. So $D|_{N_i} = 0$ for all $i \geq 1$.

Thus, we have established the following proposition.

Proposition 6.2. *The action of the Lie algebra $\mathrm{Der}(A)$ on $\overline{B}_1$, B_i for $i \geq 2$, and N_i for $i \geq 1$ factors through the image of the natural map $\psi : \mathrm{Der}(A) \to \mathrm{Der}(A/M_3)$.*

Proposition 6.3. *The map $\psi : \mathrm{Der}(A_n) \to \mathrm{Der}(A_n/M_3)$ is surjective.*

Proof. Let $\delta \in \mathrm{Der}(A_n/M_3)$ be a derivation, and let $\delta(x_i) = \alpha_i$. Let $a_i \in A_n$ be such that $\widetilde{\phi}(a_i) = \alpha_i$, and let $D = \sum a_i \frac{\partial}{\partial x_i}$, which means $D(x_i) = a_i$. Then $\psi(D) = \delta$. $\qquad\square$

So we have an action of $\mathrm{Der}(A/M_3) = \mathrm{Der}(\Omega_*^{\mathrm{even}})$ on $\overline{B}_1$, B_i for $i \geq 2$, and N_i.

Now note that $*$ is an invariant operation, so $W_n \subset \mathrm{Der}(\Omega_*^{\mathrm{even}})$. This gives an action of W_n on $\overline{B}_1, B_i, \ i \geq 2$, and $N_i, i \geq 1$, and proves Theorem 2.19.

Remark 6.4. We leave it to the reader to check that the action of W_n on $\overline{B}_1 = \Omega^{\mathrm{even}}/\Omega_{\mathrm{exact}}^{\mathrm{even}}$, $N_1 = \mathbb{C}[x_1, \cdots, x_n]$ and $N_2 = \Omega^{\mathrm{even},+}$ is the standard action.

7. Representations of W_n

Now let us discuss the representation theory of W_n.

Definition 7.1. A module M over W_n belongs to the category $\mathcal{C}$ if the operators $x_i \frac{\partial}{\partial x_i} \in W_n$ on M are semisimple with finite dimensional joint eigenspaces, and their eigenvalues are in $\mathbb{Z}_{\geq 0}$.

Example 7.2. (1) $M = \mathbb{C}[x_1, x_2, \ldots, x_n]$, with the tautological action of vector fields $v = \sum f_i(x_1, ..., x_n)\frac{\partial}{\partial x_i}$. Then M is graded by $\mathbb{Z}_{\geq 0}^n$: one has $M = \bigoplus_{i_1,\ldots,i_n \geq 0} M[i_1, \ldots, i_n]$. This is a grading by eigenvalues of $x_i \frac{\partial}{\partial x_i}$, so $M \in \mathcal{C}$.
  (2) *Tensor field modules.* This is a generalization of the previous example. Namely, let $V = (\mathbb{C}^n)^*$, and let $T_N = SV \otimes V^{\otimes N}$ be the space of tensors of type $(N, 0)$ on $V^* = \mathbb{C}^n$. This space has a basis $x_1^{m_1} \ldots x_n^{m_n} dx_{i_1} \otimes \ldots \otimes dx_{i_N}$. The vector field $v = \sum_i f_i(x_1, ..., x_n)\frac{\partial}{\partial x_i}$ acts by the Leibniz rule, as follows:
$$v(\phi dx_{i_1} \otimes \ldots \otimes dx_{i_N}) = v(\phi)dx_{i_1} \otimes \ldots \otimes dx_{i_N}$$
$$+\phi dv(x_{i_1}) \otimes dx_{i_2} \otimes \ldots \otimes dx_{i_N} + \ldots + \phi dx_{i_1} \otimes \ldots \otimes dv(x_{i_N}).$$
Note that $T_0 = \mathbb{C}[x_1, ..., x_n]$, the module from the previous example, and $T_1 = \Omega^1$, the module of 1-forms.

The modules T_N, in general, are not irreducible or even indecomposable. Namely, T_N carries an action of the symmetric group S_N which commutes with W_n, and therefore decomposes into isotypic components according to the type of the S_N-symmetry. Let us consider this decomposition in more detail.

Recall that irreducible representations of S_N are labeled by partitions of N. If λ is a partition of N, let π_λ denote the corresponding

representation of S_N. Then we can define the W_n-module

$$\mathcal{F}_\lambda := \operatorname{Hom}_{S_N}(\pi_\lambda, T_N) = SV \otimes S_\lambda(V),$$

where $S_\lambda(V) := \operatorname{Hom}_{S_N}(\pi_\lambda, V^{\otimes N})$ is the corresponding Schur functor. Clearly $\mathcal{F}_\lambda \neq 0 \iff \operatorname{length}(\lambda) \leq n$.

Example 7.3. Suppose that $\lambda = (1^N)$, $N \leq n$. Then $\mathcal{F}_\lambda = \Omega^N(\mathbb{C}^n)$. In this case $\mathcal{F}_\lambda$ is not irreducible if $N \neq n$. Indeed we have an exact sequence

$$0 \to \Omega^N_{\text{closed}} \to \Omega^N \to \Omega^{N+1}_{\text{exact}} \to 0,$$

where $\Omega^{N+1}_{\text{exact}} = \Omega^{N+1}_{\text{closed}}$. Note that $\Omega^n_{\text{closed}} = \Omega^n = \mathcal{F}_{1^n}$.

Proposition 7.4. ([R]) Ω^N_{closed} is irreducible for all $N = 0, \ldots, n$.

Theorem 7.5. ([R])
   (1) $\mathcal{F}_\lambda$ is irreducible unless $\lambda = 1^N$, $N < n$.
   (2) Any simple object in $\mathcal{C}$ is isomorphic to $\mathcal{F}_\lambda (\lambda \neq 1^N)$ or to $\Omega^N_{\text{closed}}(N = 0, \ldots, n)$, while these modules are pairwise non-isomorphic.

We showed above that $\overline{B}_1$, $B_i(i \geq 2)$ and $N_i(i \geq 1)$ are W_n- modules. It is easy to see that they belong to $\mathcal{C}$. Hence they admit a composition series with successive quotients $\mathcal{F}_\lambda$, Ω^N_{closed}, where each $\mathcal{F}_\lambda$ or Ω^N_{closed} has finite multiplicity.

Proposition 7.6. The W_n-modules $\overline{B}_1$, B_m for $m \geq 2$, and N_m have finite length.

Proof. Let $0 \neq Y \in \mathcal{C}$, and let $d_i = \dim Y[i]$, the dimension of the subspace of elements of degree i in Y. It is easy to see from Theorem 7.5 that $d_i \sim \frac{Cn^i}{i!}$ as $i \to \infty$, where $C \geq 1$ is an integer. Therefore, it suffices to show that $\dim B_m[i] \leq C_m n^i$ for some $C_m > 0$ and sufficiently large i. But this follows by induction in m using Theorem 2.12 from the case $m = 2$, which is obtained from Theorem 2.17(4). Also, the statement about $\overline{B}_1$ follows from Theorem 2.17(5).

To prove the statement for N_m, note that by Theorem 2.10, $N_m = AB_m = \sum_s A[c_s, B_{m-1}]$, where c_s runs through the elements $x_i, x_i x_j$, and $x_i[x_j, x_k]$. Now, $a[c_s, b] = [ac_s, b] - [a, b]c_s$, so we get $N_m \subset B_m + \sum_s B_m c_s$. This means that $\dim N_m[i] \leq \widetilde{C}_m n^i$ for large i (as we have a similar bound for B_m). This implies that N_m have finite length. $\square$

8. PROOF OF THEOREM 2.17(3)

8.1. **Proof of Theorem 2.17(3) for** $n = 2$. Now we will prove Theorem 2.17(3) in the special case $n = 2$. Recall that Theorem 2.17(3)

states that $L_2 \cap M_3 = L_3$, or, equivalently, the natural surjective map

$$\theta : B_2 \to \Omega^{\text{even}}_{\text{exact}}$$

is an isomorphism. In the case $n = 2$, $\Omega^{\text{even}}_{\text{exact}} = \Omega^2 = \mathbb{C}[x, y]dx \wedge dy$, which is a bigraded space with a 1-dimensional space in bidegree (i, j) for all $i, j \geq 1$ and zero everywhere else. So, since θ is surjective, it suffices to prove the following proposition.

Proposition 8.1. *If $A = A_2$ then B_2 is spanned by the (images of) the elements $[y^i, x^j]$.*

Proof. Note that in the free algebra generated by x, a, we have:
$$\begin{aligned}
[a, x^m] &= ax^m - x^m a \\
&= ax^m - xax^{m-1} + xax^{m-1} - x^2 ax^{m-2} + \ldots + x^{m-1}ax - x^m a \\
&= [ax^{m-1}, x] + [xax^{m-2}, x] + \ldots + [x^{m-1}a, x].
\end{aligned}$$
Modulo L_3 all these summands are the same, so we have

$$[a, x^m] = m[x^{m-1}a, x]$$

modulo L_3, and hence

$$[x^{m-1}a, x] = \frac{1}{m}[a, x^m]$$

modulo L_3. More generally,
$$\begin{aligned}
[x^k a, x^r] &= x^k ax^r - x^{k+r}a \\
&= x^k ax^r - x^{k+1}ax^{r-1} + x^{k+1}ax^{r-1} - x^{k+2}ax^{r-2} + \ldots \\
&\quad + x^{k+r-1}ax - x^{k+r}a = [x^{k-1}ax^r, x] + \ldots + [x^{k+r-1}a, x],
\end{aligned}$$
which modulo L_3 equals $r[x^{k+r-1}a, x]$ (since all the r summands are the same modulo L_3). So we get

$$[x^k a, x^r] = \frac{r}{k+r}[a, x^{k+r}]$$

modulo L_3.

Now we proceed to prove the proposition. Clearly, L_2 and hence B_2 is spanned by elements of the form $[a, x]$ and $[a, y]$, $a \in A$. So it suffices to show that for any $a \in A$, $[a, x]$ is a linear combination of $[y^i, x^j]$ modulo L_3 (the case of $[a, y]$ is similar).

To this end, it is enough to show that for any monomial $b \in A$ and $r \geq 1$, $[b, x^r]$ is a linear combination of $[y^i, x^j]$ modulo L_3. We will prove this statement by induction in the degree d of b. The base case ($d = 1$) is clear. To justify the step of induction, let b have degree d. If b' is obtained by cyclic permutation of b, then $[b, x] = [b', x]$ modulo L_3, so we can replace b with b'. Also, if $b = y^d$, we are done. Otherwise, there exists a cyclic permutation b' of b such that $b' = x^k a$, $k > 0$.

Then, as shown above,

$$[b, x^r] = [x^k a, x^r] = \frac{r}{k+r}[a, x^{k+r}]$$

modulo L_3. Since $\deg(a) < d$, we are done by the induction assumption. This justifies the induction step and proves the proposition. $\qquad\square$

8.2. **Proof of Theorem 2.17(3) for all** n.

Lemma 8.2. *Let* $\zeta : \Omega \otimes \mathbb{C}^n \to \Omega$ *defined by* $\zeta(\sum_i \alpha_i \otimes e_i) = \sum_i d\alpha \wedge dx_i$ *where* e_i *is the standard basis of* $\mathbb{C}^n$. *Then* $\ker\zeta$ *is spanned by the following:*

(1) $b \otimes e_i$, *where* $b \in \Omega_{\text{closed}}$;
(2) *Elements* $\beta \wedge dx_i \otimes e_i$ *and* $\beta \wedge dx_j \otimes e_i + \beta \wedge dx_i \otimes e_j$, $\beta \in \Omega$;
(3) *Elements* $\sum_i \frac{\partial f}{\partial x_i} \otimes e_i$, $f \in \Omega^0$.

Proof. It is easy to see that elements of types (1),(2),(3) are contained in the kernel of ζ. Let us now prove that any element in the kernel of ζ is a linear combination of elements of the form (1),(2),(3). Let $\alpha = \sum_i \alpha_i \otimes e_i \in \ker\zeta$. We may assume that α is homogeneous (i.e. $\alpha \in \Omega^k \otimes \mathbb{C}^n$ for some k). We have $\zeta(\alpha) = \sum_i d\alpha_i \wedge dx_i = 0$, so $w = \sum_i \alpha_i \wedge dx_i$ is closed and thus w is exact. Let us first assume that $k > 0$. Then by the Poincaré lemma, we have $w = d(\sum_i b_i \wedge dx_i)$ for some $b_i \in \Omega$. Then $\sum_i (\alpha_i - db_i) \wedge dx_i = 0$. Therefore, by subtracting from α elements of type (1), we can replace $\alpha_i - db_i$ by α_i and assume without loss of generality that $\sum_i \alpha_i \wedge dx_i = 0$. By standard linear algebra (exactness of the Koszul complex), this means that $\alpha_i = \sum_j \alpha_{ij} \wedge dx_j$, $\alpha_{ij} = \alpha_{ji}$. So

$$\alpha = \sum_{i,j} \alpha_{ij} \wedge dx_j \otimes e_i = \sum_i \alpha_{ii} \wedge dx_i \otimes e_i + \sum_{i<j} (\alpha_{ij} \wedge dx_j \otimes e_i + \alpha_{ij} \wedge dx_i \otimes e_j).$$

This is a linear combination of elements of type (2), as desired.

Now assume that $k = 0$. Then $\sum_i d\alpha_i \wedge dx_i = 0$, so the 1-form $\sum \alpha_i dx_i$ is closed. Hence it is exact, which means $\alpha_i = \frac{\partial f}{\partial x_i}$ for some $f \in \Omega^0$, i.e. α is an element of type (3). $\qquad\square$

Lemma 8.3. *In* A_4, *the images of* $[x_1[x_2, x_3], x_4] \in B_2$ *are antisymmetric in* x_1, x_2, x_3, x_4.

Proof. The antisymmetry in x_2, x_3 is clear. We show the antisymmetry in x_1, x_2.

$$
\begin{aligned}
[x_1[x_2, x_3], x_4] &= [[x_1 x_2, x_3], x_4] - [[x_1, x_3]x_2, x_4] \\
&= [[x_1 x_2, x_3], x_4] - [x_2[x_1, x_3], x_4] + [[x_2, [x_1, x_3]], x_4] \\
&= -[x_2[x_1, x_3], x_4] \mod L_3.
\end{aligned}
$$

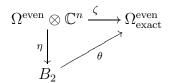Now we show the antisymmetry in x_1, x_4.

$$
\begin{aligned}
[x_1[x_2, x_3], x_4] &= [x_1, x_4][x_2, x_3] + x_1[[x_2, x_3], x_4] \\
&= [x_1, x_4][x_2, x_3] + [[x_2, x_3], x_1 x_4] - [[x_2, x_3], x_1]x_4 \\
&= [x_1, x_4][x_2, x_3] - x_4[[x_2, x_3], x_1] \\
&= -[x_4[x_2, x_3], x_1] \mod L_3.
\end{aligned}
$$
$\square$

Now we prove Theorem 2.17(3). Recall that L_2 is spanned by elements of the form $[a, x_i]$, $a \in A$. Also, we know from Lemma 3.1 that $[M_3, A] \subset L_3$. Thus we have a surjective map $\eta : A/M_3 \otimes \mathbb{C}^n \to B_2$ defined by $\eta(a \otimes e_i) = [a, x_i]$. This means that we have a surjective map $\eta : \Omega^{\text{even}} \otimes \mathbb{C}^n \to B_2$. Then we have the following diagram

$$
\begin{array}{ccc}
\Omega^{\text{even}} \otimes \mathbb{C}^n & \xrightarrow{\ \zeta\ } & \Omega^{\text{even}}_{\text{exact}} \\
{\scriptstyle\eta}\downarrow & \nearrow_{\theta} & \\
B_2 & &
\end{array}
$$

which is commutative. So to prove the result, it suffices to show that $\ker\zeta = \ker\eta$. Thus we have to show that any element of $\ker\zeta$ is contained in $\ker\eta$. By Lemma 8.2, it suffices to check that elements of type (1) and (2) are killed by η.

Type (1) is clear, since if $a \in A/M_3$ maps to $\Omega^{\text{even}}_{\text{exact}}$, then $a \in L_2$ as we showed in Corollary 5.4 (i).

Now we show this for type (2). First we show that

$$
\eta(\beta dx_i \otimes e_i) = 0 \ \forall \beta.
$$

To this end, take $a = b[x_k, x_i]$. We have $\phi(b[x_k, x_i]) = \phi(b) \wedge dx_k \wedge dx_i$. Thus we have $\eta(\phi(b) \wedge dx_k \wedge dx_i \otimes e_i) = [b[x_k, x_i], x_i] = 0 \mod L_3$ since it is antisymmetric in the last two variables by Lemma 8.3. But each β is a linear combination of $\phi(b) \wedge dx_k$.

Now deal with $\eta(\beta dx_i \otimes e_j + \beta dx_j \otimes e_i)$ similarly. Take $a = b[x_k, x_j]$. We have $\phi(b[x_k, x_j]) = \phi(b) \wedge dx_k \wedge dx_j$, so

$$
\eta(\phi(b) \wedge dx_k \wedge dx_i \otimes e_j + \phi(b) \wedge dx_k \wedge dx_j \otimes e_i) =
$$

$$
[b[x_k, x_i], x_j] + [b[x_k, x_j], x_i] = 0 \mod L_3,
$$

again by Lemma 8.3.

It remains to consider the case of elements of type (3). It is enough to take $f = (\sum_j \lambda_j x_j)^m$, $\lambda_i \in \mathbb{C}$, $m \in \mathbb{Z}_{\geq 1}$, since such elements along

with constants span Ω^0. Then $\frac{\partial f}{\partial x_i} = m\lambda_i(\sum_j \lambda_j x_j)^{m-1}$. So the corresponding element of type (3) is

$$\alpha = m(\sum_j \lambda_j x_j)^{m-1} \otimes \sum_i \lambda_i e_i$$

Thus $\eta(\alpha) = m[(\sum_j \lambda_j x_j)^{m-1}, \sum_i \lambda_i x_i] = 0$, as needed.

9. Proof of Theorem 2.10

Lemma 9.1. *([AJ], Lemma 3.1) One has*

$$[u^3, [v, w]] = 3[u^2, [uv, w]] - 3[u, [u^2v, w]] + \frac{3}{2}[u^2, [v, [u, w]]] - \frac{3}{2}[u, [v, [u^2, w]]] +$$

$$+ [u, [u, [u, [v, w]]]] - \frac{3}{2}[u, [u, [v, [u, w]]]] + \frac{3}{2}[u, [v, [u, [u, w]]]].$$

Proof. Direct computation (by hand or using a computer). □

Corollary 9.2. *([AJ], Corollary 3.2) Let $a, b, c \in A$, and*

$$S(a, b, c) = \frac{1}{6}(abc + acb + bac + bca + cab + cba)$$

be the average of products of a, b, c in all orders. Then for $m \geq 2$

$$[S(a, b, c), B_m] \subset [ab, B_m] + [bc, B_m] + [ca, B_m] + [a, B_m] + [b, B_m] + [c, B_m]$$

inside B_{m+1}.

Proof. In Lemma 9.1, set $u = t_1 a + t_2 b + t_3 c$, v to be any element of A, and w to be any element of L_{m-1}, and take the coefficient of $t_1 t_2 t_3$. □

Now we are ready to prove the theorem. We may assume that $A = A_n$, generated by $x_1, ..., x_n$. Let P be a monomial in x_i. We have to show that $[P, B_m]$ is contained in the sum of $[x_i, B_m]$, $[x_i x_j, B_m]$, and $[x_i[x_j, x_k], B_m]$ (the latter with i, j, k distinct). Since by Proposition 4.2, $[M_3, L_j] \subset L_{j+2}$, we may view P as an element of A/M_3. By Theorem 2.17, $A/M_3 \cong \Omega_*^{\mathrm{even}}$, so we may view P as an element of Ω_*^{even}.

Let E be the linear span of elements of the form $S(a, b, c)$, where $a, b, c \in \Omega_*^{\mathrm{even}}$ are of positive degree. Let X be the span of $1, x_i, x_i x_j$ and $x_i dx_j \wedge dx_k$ with i, j, k distinct.

Lemma 9.3. *([AJ]) One has $\Omega_*^{\mathrm{even}} = E + X + \Omega_{\mathrm{exact}}^{\mathrm{even}}$.*

Proof. Note that it suffices to check this for the associated graded algebra of Ω_*^{even} under the filtration by rank of forms, i.e. for Ω^{even} under ordinary multiplications. Note that Ω^{even} is a commutative algebra, so in this algebra $S(a, b, c) = abc$. Also, Ω^{even} is generated by x_i and

$dx_j \wedge dx_k$. Thus, Ω^{even}/E is spanned by $1, x_i, x_ix_j, x_i dx_j \wedge dx_k, dx_j \wedge dx_k$, and $dx_i \wedge dx_j \wedge dx_k \wedge dx_l$. But the last two forms are exact, as is $x_i dx_i \wedge dx_k$. This implies the lemma. $\qquad\square$

Now we prove by induction in degree of $P \in \Omega_*^{\text{even}}$ that

$$[P, B_m] \subset \sum_i [x_i, B_m] + \sum_{i,j} [x_ix_j, B_m] + \sum_{i,j,l} [x_i[x_j, x_k], B_m].$$

The base of induction is obvious. The induction step follows from Lemma 9.3, Theorem 2.17(4) and Corollary 9.2. This implies the theorem.

10. Application of representation theory of W_n to the study of B_i and N_i and proof of Theorem 2.13

Let us explain how the representation theory of W_n can be used to study B_i and N_i.

Proposition 10.1. *If $i \geq 3$ then $\Omega_{\text{closed}}^N (0 \leq N \leq n)$ does not occur as a composition factor in B_i nor N_i.*

Proof. First, consider $N = n$, and look at the polylinear part in x_i. In A_n this part is the regular representation of S_n, so contains a single copy of the sign representation of S_n. But this copy already occurs in $\overline{B}_1 \oplus B_2$ and $N_1 \oplus N_2$, so does not occur in $\underset{i \geq 3}{\oplus} B_i$ nor $\underset{i \geq 3}{\oplus} N_i$. Hence, Ω_{closed}^N cannot occur in these, as it contains such a copy.

Now, for $N < n$, Ω_{closed}^N still cannot occur, as it contains a vector $dx_1 \wedge \cdots \wedge dx_N$ in degree $(\underbrace{1, \cdots, 1}_{N}, 0 \cdots, 0)$, which leads to a contradiction with the above if we mod out the other variables $x_{N+1}, \cdots, x_n$. $\quad\square$

Thus, B_i, N_i $(i \geq 3)$ are equal in the Grothendieck group of $\mathcal{C}$ to $\oplus_\lambda \mathcal{F}_\lambda$. So, for the multivariable Hilbert series (in which the power of the variable t_i counts the degree with respect to x_i) we have

$$h_{B_i}(t_1, \cdots, t_n) = \sum_\lambda m_\lambda h_{\mathcal{F}_\lambda}(t_1, \cdots, t_n) = \frac{\sum_\lambda m_\lambda h_{S_\lambda(V)}(t_1, \cdots, t_n)}{(1 - t_1) \cdots (1 - t_n)}.$$

An analogous formula holds for N_i. Moreover, by Theorem 7.6, B_i and N_i have finite length, i.e. the sum over λ is finite, i.e. the numerator is a polynomial. This proves Theorem 2.13 (by setting $t_i = t$ for all i).

For $n = 2$ we know from [AJ] that $|\lambda| \leq 2m - 3$ for B_m and from [Ke] that $|\lambda| \leq 2m - 2$ for N_m. Therefore,

$$h_{B_i} = \frac{\sum_{\lambda} m_\lambda (t_1^{\lambda_1} t_2^{\lambda_2} + t_1^{\lambda_1 - 1} t_2^{\lambda_2 + 1} + \cdots + t_1^{\lambda_2} t_2^{\lambda_1})}{(1 - t_1)(1 - t_2)},$$

with $|\lambda| \leq 2m - 3$, and h_{N_i} is given by a similar formula with $|\lambda| \leq 2m - 2$.

This means that it suffices to know h_{B_i} for degrees $\leq 2m - 3$ and h_{N_i} for degrees $\leq 2m - 2$. This allows one to compute $h_{B_i} (i \leq 7)$ by using computer, which is done in [AJ].

11. LOWER CENTRAL SERIES OF ALGEBRAS WITH RELATIONS

Consider now the lower central series of algebras with relations. In this case, much less is known than for free algebras, and we will discuss some theoretical and experimental results and conjectures. Specifically, consider the algebra $A := \mathbb{C} < x, y > / < P >$, where P is a noncommutative polynomial of x, y of some degree d with square-free abelianization (i.e. the corresponding commutative polynomial factors into distinct linear factors). In this case, A_{ab} is the function algebra on a union of d lines, so by Theorem 2.21, B_m and N_m are finite dimensional for $m \geq 2$. This means that $B_m[r] = 0$ and $N_m[r] = 0$ for sufficiently large r. The following theorem gives a bound for how large r should be.

Proposition 11.1. *(i) For $m \geq 2$ we have $B_m[r] = 0$ if $r \geq 2d + 2m - 5$.*
(ii) For $m \geq 2$ we have $N_m[r] = 0$ if $r \geq 2d + 2m - 4$.

Proof. (i) By [KL], Theorem 1, the Hilbert series of B_2 is $h_{B_2}(t) = t^2(1 + t + ... + t^{d-2})^2$. The degree of this polynomial is $2d - 2$, so we have $B_2[r] = 0$ for $r \geq 2d - 1$. Now, by Theorem 1.3(4) of [AJ] (Theorem 2.12 for $m = 2$), we have

$$B_{m+1} = [x, B_m] + [y, B_m] + [xy, B_m].$$

This implies that if $B_m[r] = 0$ for $r \geq s$ then $B_{m+1}[r] = 0$ if $r \geq s + 2$. Thus, arguing by induction starting from $m = 2$, we get that $B_m[r] = 0$ for $r \geq 2d - 1 + 2(m - 2) = 2d + 2m - 5$.

(ii) This follows from (i) and Theorem 1.2 of [Ke], which implies that $N_i = xB_i + yB_i$. $\qquad\square$

Now consider the case when P is "Weil generic", i.e. "outside of a countable union of hypersurfaces" (in the space of noncommutative polynomials of degree d). In this case, it is clear from Chevalley's constructibility theorem that $\dim B_m[r]$ and $\dim N_m[r]$ are independent

of P. Proposition 11.1 says that the width of the interval of nonzero values for $\dim B_m[r]$ is at most $2d + m - 5$ (these values may occur for $m \leq r \leq 2d + 2m - 6$). However, computer calculations for small d show that this bound is not sharp, and the width does not actually increase with m for fixed d; for instance, it appears to be ≤ 3 for $d = 3$ and ≤ 5 for $d = 4$. For this reason, on the basis of computational evidence we make the following conjecture:

Conjecture 11.2. For Weil generic P of degree d, we have $B_m[r] = N_m[r] = 0$ for $r \geq 2d + m - 3$. In other words, the width of the interval of nonzero values is $\leq 2d - 3$.

For instance, for $d = 3$ (the smallest nontrivial case), the prediction is that $B_m[r] = 0$ for $r \geq m + 3$.

Now consider the structure of B_m in more detail. Let FL_2 be the free Lie algebra in generators x, y. We have a natural Lie algebra homomorphism $\widetilde{\psi} : FL_2 \to A$. Observe that $B_m[m] = L_m[m] = \widetilde{\psi}(FL_2[m])$. In degree ≥ 2, the image of this homomorphism is contained in $[A, A]$. Thus, we have a homomorphism of Lie algebras $\psi : FL_2^{\geq 2} \to [A, A]$, whose image in degree m is $L_m[m] = B_m[m]$.

Now, the Hilbert series of $FL_2^{\geq 2}$ and of $[A, A]$ can be computed explicitly. For the former, it is obtained in a standard way from the PBW theorem:

$$h_{FL_2^{\geq 2}}(t) = \sum_{i \geq 2} a_i t^i, \text{ where } \prod_{i \geq 2}(1 - t^i)^{a_i} = \frac{1 - 2t}{(1 - t)^2}.$$

For the latter, we have

$$h_{[A,A]}(t) = \sum_{i \geq 2} c_i t^i = h_A(t) - h_{A/[A,A]}(t),$$

and we have

$$h_A(t) = \frac{1}{1 - 2t + t^d}$$

(see [EG], Theorem 3.2.4; the term t^d accounts for the relation of degree d), while

$$h_{A/[A,A]}(t) = 1 + \sum_{i \geq 1} b_i t^i, \text{ where } \prod_{i=1}^{\infty}(1 - t^i)^{b_i} = \prod_{s=1}^{\infty}(1 - 2t^s + t^{ds})$$

([EG], Theorem 3.7.7). This implies that $\lim_{n \to \infty} a_n^{1/n} = 2$, while $\lim_{n \to \infty} c_n^{1/n} = \delta^{-1}$, where δ is the smallest positive root of the equation $1 - 2t + t^d = 0$ (clearly, $\delta^{-1} < 2$). So we have that $c_n < a_n$ for large enough n, hence ψ is not injective starting from some degree. This gives rise to the following question.

Question 11.3. Is ψ surjective in some degree m?

Note that surjectivity of ψ in two consecutive degrees implies strong consequences about the structure of the lower central series. Namely, we have the following proposition.

Proposition 11.4. *Let $A = A_n(\mathbb{C})/I$, where I is a homogeneous ideal. Let $q \geq 3$, and suppose that $B_2[m] = 0$ for $m \geq q$, and for some $m \geq q+1$, the natural map $\psi : FL_n^{\geq 2} \to [A, A]$ is surjective in degrees m and $m - 1$. Then $[A, A][\ell] = L_\ell[\ell] = B_\ell[\ell]$ and $B_s[\ell] = 0$ for $2 \leq s \leq \ell - 1$ for all $\ell \geq m - 1$.*

Proof. Note that if ψ is surjective in some degree $m \geq 3$ then $[A, A][m] = L_m[m]$, so in particular $B_i[m] = 0$ for $i = 2, ..., m - 1$, and we have $[z_{m-2}, ..., z_1, x_i x_j] \in L_m[m]$ if each z_p is one of the generators x_i. Hence, using [BJ], Corollary 1.5 (Theorem 2.12 above), we have

$$B_m[m + 1] = \sum_i [x_i, B_{m-1}[m]] + \sum_{i,j} [x_i x_j, B_{m-1}[m - 1]] =$$

$$= \sum_{i,j} [x_i x_j, B_{m-1}[m - 1]],$$

since $B_{m-1}[m] = 0$. Now, using Lemma 3.3, we can put $x_i x_j$ in the innermost slot, and get that $B_m[m + 1]$ is spanned by linear combinations of $[z_{m-1}, ..., z_1, x_i x_j]$, where each z_p is one of the generators x_i. But as explained above, $[z_{m-2}, ..., z_1, x_i x_j] \in L_m[m]$, so we get that $B_m[m + 1] = 0$.

Now, if ψ is also surjective in degree $m - 1$, then we get $B_i[m-1] = 0$ for $i = 2, ..., m - 2$, so for any $3 \leq i \leq m - 1$, we have (using [BJ], Corollary 1.5 again):

$$B_s[m + 1] = \sum_i [x_i, B_{s-1}[m]] + \sum_{i,j} [x_i x_j, B_{s-1}[m - 1]] = 0.$$

So we get that if $m \geq q+1$ and ψ is surjective in degrees m and $m - 1$ then it is also surjective in degree $m + 1$ (and thus by induction in all degrees $\ell \geq m - 1$). This implies the statement. $\square$

We don't know, however, if the answer to Question 11.3 is positive, and in fact this hope is not supported by computational evidence. Let us consider the case $d = 3$, and $A = A_2/ < P >$, where P is a Weyl generic element of degree 3. In this case, the smallest degree where ψ has a chance of being surjective (i.e. the smallest m for which $a_m \geq c_m$) is $m = 16$. Namely, we have $a_{16} = 4080$, while $c_{16} = 4036$. Computation shows, however, that the map ψ in degree 16 is not surjective (even though surjectivity is possible dimensionwise): it

has rank 4031 and a 5-dimensional cokernel. A similar pattern occurs in degrees $17, 18, 19$: while $a_n > c_n$ in these degrees, the map ψ has a nonzero cokernel of dimensions $4, 5, 4$. Namely, we have:

$$a_{17} = 7710, \ c_{17} = 6552, \ \text{rank}\psi[17] = 6548;$$

$$a_{18} = 14532, \ c_{18} = 10615, \ \text{rank}\psi[18] = 10610;$$

$$a_{19} = 27594, \ c_{19} = 17216, \ \text{rank}\psi[19] = 17212.$$

Thus, instead of eventually vanishing, the dimensions of the cokernels of ψ seem to stabilize to the pattern $5, 4, 5, 4, ...$ We do not know an explanation for this phenomenon, and do not know if it continues beyond degree 19.

We note, however, that according to our computations, it appears that ψ is eventually surjective (i.e. we are in the setting of Proposition 11.4) in the case when $A = A_2/ < P, Q >$, where P is Weil generic of degree 3 and Q is Weil generic of degree 8. In this case, the smallest degree in which surjectivity of ψ is possible dimensionwise is 15, and a computer calculation shows that ψ is indeed surjective in degrees 15 and 16 (of ranks 1974, 3045, respectively). Thus Proposition 11.4 applies for degrees $m \geq 15$. In lower degrees, the dimensions of $B_m[i]$ can be easily computed by a computer algebra system; thus, one can get a complete list of dimensions of $B_m[i]$ in this case.

Remark 11.5. Note that 8 is the smallest integer n such that the series $(1 - 2t + t^3 + t^n)^{-1}$ has positive coefficients, and therefore the algebra A is infinite dimensional by the Golod-Shafarevich inequality, [GS].

Remark 11.6. These computations were done using a MAGMA program written by Eric Rains. It computes over a large finite field with randomly chosen relations of the given degrees. Thus, the computational results of this subsection should be viewed as conjectural.

REFERENCES

[AJ]   N. Arbesfeld, D. Jordan, New results on the lower central series quotients of a free associative algebra, Journal of Algebra, v. 323, Issue 6, 2010, 1813-1825, arXiv:0902.4899.

[BB]   M. Balagovic, A. Balasubramanian, On the Lower Central Series Quotients of a Graded Associative Algebra, Journal of Algebra, 328, Issue 1, 2011, 287-300. arXiv:1004.3735.

[BJ]   A. Bapat, D. Jordan, Lower central series of free algebras in symmetric tensor categories, Journal of Algebra, v. 373, 2013, 299-311, arXiv:1001.1375.

[BEJKL] S. Bhupatiraju, P. Etingof, D. Jordan, W. Kuszmaul, and J. Li, Lower central series of a free associative algebra over the integers and finite fields, v. 372, 2012, 251-274, arXiv:1203.1893.

[BoJ]   B. Bond, D. Jordan, The lower central series of the symplectic quotient of a free associative algebra, J. pure and applied algebra, 217 (4), 689-699. arXiv:1111.2316.

[CFZ]   K. Cordwell, T. Fei, K. Zhou, On Lower Central Series Quotients of Finitely Generated Algebras over $\mathbb{Z}$, Journal of Algebra, 423, 2015, 559-572, arXiv:1309.1237.

[DE]   G. Dobrovolska, P. Etingof, An upper bound for the lower central series quotients of a free associative algebra, International Mathematics Research Notices, 2008, rnn039, arXiv:0801.1997.

[DKM]   G. Dobrovolska, J. Kim, X. Ma, On the lower central series of an associative algebra (with an appendix by Pavel Etingof), J. Algebra, 320, 2008, no. 1, 213-237, arXiv:0709.1905.

[EG]   P. Etingof, V. Ginzburg, Noncommutative complete intersections and matrix integrals, Pure Appl. Math. Q., 3 (1), 2007, pp. 107151, arXiv:math/0603272.

[EKM]   P. Etingof, J. Kim, and X. Ma, On universal Lie nilpotent associative algebras, Journal of Algebra, 321.2, 2009, 697-703, arXiv:0805.1909.

[FS]   B. Feigin, B. Shoikhet, On $[A, A]/[A, [A, A]]$ and on a W_n-action on the consecutive commutators of free associative algebra, Mathematical Research Letters 2007; 14(5), pp.781-795, arXiv:0610410.

[FX]   Y. Fregier, I. Xia, Lower Central Series Ideal Quotients Over F_p and $\mathbb{Z}$. arXiv:1506.08469.

[GS]   E. Golod, I. Shafarevich, (1964), On the class field tower, Izv. Akad. Nauk SSSR 28: 261272 (in Russian).

[GL]   N. Gupta, F. Levin, On the Lie ideals of a ring, Journal of Algebra, v.81, issue 1, 1983, 225-231.

[J]   S. Jennings, On rings whose associated Lie rings are nilpotent, Bulletin of the American Mathematical Society, v.53, issue 6, 1947, 593-597.

[JO]   D. Jordan and H. Orem, Algebro-geometric Construction of Lower Central Series, International Mathematics Research Notices, rnu125, 2014, doi:10.1093/imrn/rnu125, arXiv:1302.3992.

[KL]   L. Kendrick, A. Lonergan, The Lower Central Series of the Quotient of a Free Algebra, arXiv:1503.01447.

[Ke]   G. Kerchev, On the filtration of a free algebra by its associative lower central series, arXiv:1101.5741.

[Kr]   A. Krasilnikov, The additive group of a Lie nilpotent associative ring, Journal of Algebra, 392, 2013, 10-22.

[R]   A. Rudakov, Irreducible representations of infinite-dimensional Lie algebras of Cartan type, Izvestiya: Mathematics v.8, issue 4, 1974, 836-866.

Mathematical Sciences Department Princess Nourah bint Abdul-rahman University P.O.Box 84428, Riyadh 11671, Saudi Arabia
E-mail address: nhabughazala@pnu.edu.sa

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: etingof@math.mit.edu